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On Generalized Sasaki Projections†

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Let L be an orthoalgebra, and let $\mathfrak{F}(L)$ be the complete lattice of filters on L. We describe a natural mapping Δ: $L \times \mathfrak{F}(L) \to \mathfrak{F}(L)$ that specializes to the familiar Sasaki map in the case that L is an orthomodular lattice. The mapping Δ is related to the generalized Sasaki map of Bennett and Foulis. The two mappings are essentially the same if L is an orthomodular poset, but can be quite different even for rather well-behaved orthoalgebras.

1. INTRODUCTION

If L is any orthocomplemented lattice, we may define a mapping ϕ: $L \times L \to L$ by $\phi(a, b) := a \wedge (a' \vee b)$. If L is Boolean, of course, $\phi(a, b)$ is simply $a \wedge b$. L is orthomodular iff, for all $a, b \in L$, $b, \leq a \Rightarrow \phi(a, b) = b$. In this context, ϕ is usually called the *Sasaki map*, and the mapping ϕ_a: $L \to L$ taking b to $\phi(a, b)$ is the *Sasaki projection* associated with $a \in L$. Sasaki projections play a crucial role in the theory of orthomodular lattices. For instance, they are exactly the closed projections in the Foulis semigroup of L which they generate [6].

Orthomodular lattices have been generalized successively to orthomodular posets, orthoalgebras, and, most recently, to effect algebras. The purpose of this note is to point out a natural extension of the Sasaki map to these more general contexts. This is related to the generalized Sasaki map introduced by Bennett and Foulis [1]. Indeed, the two are essentially the same if E is an orthomodular poset. However, simple examples show that the two maps may be quite different even for very simple and well-behaved orthoalgebras.

†This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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In addition to being quite natural from a purely mathematical viewpoint, the generalized Sasaki map described here has certain heuristic merits, even in the familiar context of orthoalgebras.

Section 2 outlines a notion of conditioning in the setting of the Foulis–Piron–Randall formalism of test spaces, supports, and "entities" [2–5], with which I shall assume that the reader is familiar. In Section 3 this is recast in purely algebraic terms in a way that generalizes to effect algebras and connects in a straightforward way with the generalized Sasaki map of Bennett and Foulis.

2. THE CONDITIONING MAP

In this section, L is an orthoalgebra and $(X, \mathfrak{A})$ is an algebraic test space generating L as its logic. Thus, L consists of equivalence classes $p(A)$ of events A of $\mathfrak{A}$ under perspectivity.

Let Σ be a collection of supports of $\mathfrak{A}$, and $\mathscr{L}$ is the associated complete lattice of properties. If $p \in L$, we write Σ_p for the collection of all $S \in \Sigma$ such that, for any representative event $A \in p$ and any test E with $A \subseteq E$, $S \cap E \subseteq A$. Thinking of S as the set of outcomes that are "possible" in some state of the entity in question, Σ_p represents the set of states in which p is certain to be confirmed if tested. The *canonical mapping* $[\cdot]$: $L \to \mathfrak{L}$ is defined by

$$[p] = \bigcup \Sigma_p$$

Note that $[\cdot]$ depends tacitly on Σ.

Each equivalence class $p = p(A)$ in the logic L, construed as a test space in its own right, can be shown to be algebraic, with logic isomorphic to the interval $[0, p]$ in L. We write X_p for the set of outcomes of this test space, i.e., $X_p = \cup_p$. It is not difficult to see that, if S is a support of $\mathfrak{A}$, then $S \cap X_p$ is a support of p. This suggests the following construction.

Definition 1. For $p \in L$ and $S \in \mathscr{L}$, let $\mathscr{L}_{p,S}$ denote the collection of all properties $T \in \mathscr{L}$ such that $T \subseteq [p]$ and $T \cap X_p \subseteq S$. We define the *conditioning map* γ_p: $\mathscr{L} \to \mathscr{L}$ by

$$\gamma_p(S) := \bigcup \mathscr{L}_{p,S}$$

To motivate this, let $\Sigma_{p,S} = \Sigma \cap \mathscr{L}_{p,S}$. Notice that $\mathscr{L}_{p,S}$ is the complete sublattice of the interval $[0, [p]]$ in $\mathscr{L}$ generated by $\Sigma_{p,S}$ and that $\gamma_p(S) = \cup \Sigma_{p,S}$. The maps $p, S \mapsto \Sigma_{p,S}$ and $p, S \mapsto \gamma_{p,S}$ represent a simple form of *conditioning*. If we are given data from a large number of tests of $p \in L$, all confirming p, and if the actual state of the entity for all of these tests was

S, then all our data lie in $X_p \cap S$. We will be inclined to infer not only that p is certain, but that the state of the entity belongs to $\Sigma_{p,S}$, and that the property $\gamma_p(S)$ is actual.

Example 1. Let $X = \{a, x, b, y, c, z\}$ and $\mathfrak{A} = \{\{a, x, b\}, \{b, y, c\}, \{c, z, a\}\}$ (the so-called "Wright triangle"), a Greechie diagram of which is given in Fig. 1. We shall compute $\gamma_p(S)$ for $p = b'$ and $S = [z] = \{b, z, y, z\}$, where Σ consists of all supports of X. Note that $X_{b'} = \{a, x, y, c\}$ and $[b'] = \{a, x, y, z, c\}$. Hence, $S \cap X_{b'} = \{x,y\}$. The largest support contained in $[b']$ having this same intersection with $X_{b'}$ is the support $\{x, y, z\}$. Hence, $\gamma_{b'}([z]) = \{x, y, z\}$.

Example 2. To illustrate the dependence of γ_a on Σ, let $\mathfrak{A}$ be as above, but suppose that Σ consists only of the principal properties $[p] = X - p^{\perp}$, where p is an atom of L. Again, $S = [z]$, $S \cap X_{b'} = \{x,y\}$, and $[b'] = \{a, x, y, z, c\}$. However, in this case the only elements of Σ below $[b']$ are $[x] = \{x, y, z, c\}$, $[y] = \{a, x, y, z\}$, $[a] = \{a, y\}$, and $[c] = \{x,c\}$. None of these has intersection with $X_{b'} = \{a, x, y, c\}$ contained in $\{x,y\}$; hence, in this setting, $\gamma_{b'}([z]) = 0$.

There is an alternative formulation of γ that is in some ways more appealing:

Theorem 1. For any $p \in L$ and $S \in \mathscr{L}$, $\gamma_p(S) = \bigwedge_{A \in p} [S \cap A]$.

Proof. Suppose that $T \in \mathscr{L}_{p,S}$. Then for every event $A \in p$, $T \cap A \subseteq S \cap A$ (since $T \cap X_p \subseteq S \cap X_p$ and $A \subseteq X_p$). Since $T \subseteq [p]$, we have for every test E containing A that $T \cap E \subseteq A$, so $T \cap E \subseteq T \cap A \subseteq S \cap A$. Hence, $T \in \Sigma_{S \cap \mathrm{A}}$, whence $T \subseteq [S \cap A]$. It follows that $\gamma_p(S) = \cup \mathscr{L}_{p,S} \subseteq [S \cap A]$.

Now suppose $T \subseteq [S \cap A]$ for every $A \in p$. Then in particular, since $[S \cap A] \subseteq [A] = [p]$, we have $T \subseteq [p]$. Now, noting that for any test E containing A we have $T \cap A = T \cap E \subseteq S \cap A$, it follows that

$$T \cap X_p = \bigcup_{A \in p} T \cap A \subseteq \bigcup_{A \in p} S \cap A = S \cap X_p$$

Hence, $T \in \mathscr{L}_{p,S}$, so $T \subseteq \gamma_p(S)$. ∎

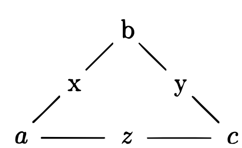

Fig. 1.

Remark. Since the principal properties are meet-dense in $\mathscr{L}$, we may extend γ to a mapping γ: $\mathscr{L} \times \mathscr{L} \to \mathscr{L}$ via

$$\gamma_P(S) = \bigwedge_{P \leq [p]} \gamma_p(S) = \bigwedge\{[S \cap A] | P \in \Sigma_A\}$$

It is not entirely obvious at this point that γ is really a generalization of the Sasaki map. For the record:

Theorem 2. Let L be an OML and let $\mathfrak{A}$ be the test space of orthopartitions of the unit in L. Let Σ consist of all supports of $\mathfrak{A}$. Then $\forall a, b \in L$, $\gamma_a([b]) = [\phi(a, b)]$.

This will follow easily from the results of the next section.

3. FORMULATION IN TERMS OF IDEALS AND FILTERS

Recall [3] that an *ideal* in an orthoalgebra (or effect algebra) L is a set $J \subseteq L$ such that $\forall a, b \in L$, $a \oplus b \in J \Leftrightarrow a, b \in J$. Dually, a *filter* on L is a set of the form $F = J' := \{x' | x \in J\}$. Note that, while ideals are lower sets in the natural ordering on L, the principal order ideal $a\downarrow := \{x \in L | x \leq a\}$ is an orthoalgebra filter for every $a \in L$ iff L is an OMP [3]; and of course, a dual result holds for principal order filters.

If A is any subset of L, we denote by (A) generated by A, i.e., the smallest ideal of L containing A, and by $\langle A \rangle$, the filter generated by A. For $A = \{a\}$, we write (a) and $\langle a \rangle$ rather than $(\{a\})$ and $\langle \{a\} \rangle$.

Let $\mathfrak{A}_L$ consist of the finite partitions of unity of L. Then supports of $\mathfrak{A}_L$ are exactly the complements of ideals of L [3, 4].

Lemma 1. If A is an event of $\mathfrak{A}_L$ with $a = \oplus A$, $[A] = [a] = L \backslash (a')$.

Proof. Let S be a support and $J = L \backslash S$ the corresponding ideal. Then, for any $E \in \mathfrak{A}_L$ with $A \subseteq E$, we have $S \cap E \subseteq A$ iff $J \cap A = \emptyset$ iff $E \backslash A \subseteq J$ iff $\oplus(E \backslash A) = a' \in J$. Hence,

$$(a') = \bigcap\{J | a' \in J\} = L \backslash \bigcup\{S | a \in S\} = L \backslash [a]. \quad \blacksquare$$

We shall now reformulate the conditioning map of Section 2 in terms of ideals and filters. Let $\mathfrak{I}(L)$ and $\mathfrak{F}(L)$ denote, respectively, the lattice of ideals and the lattice of filters of L. We define mappings

$$\Gamma: \quad L \times \mathfrak{I}(L) \to \mathfrak{I}(L) \qquad \text{and} \qquad \Delta: \quad L \times \mathfrak{F}(L) \to \mathfrak{F}(L)$$

by $\Gamma_a(I) := L \backslash \gamma_a(L \backslash I)$ and $\Delta_a(F) = \Gamma_a(F')$.

Lemma 2. Let $I, F \subseteq L$ be any ideal and any filter, respectively, of L. Then for all $a \in L$:

(a) $\Gamma_a(I) = (\{y \in L | y' \leq a \;\&\; a - y' \in I\})$.
(b) $\Delta_a(F) = \langle\{y \leq a | (a - y' = y \oplus a' \in F\}\rangle$.

Proof. (a) Let S, T be supports of $\mathfrak{A}_L$ and let $I = L \backslash S$ and $J = L \backslash T$ be the corresponding ideals. Then $T \subseteq [a]$ iff $(a') \subseteq J$, i.e., iff $a' \in J$. Also, $T \cap X_a \subseteq S \cap X_a$ iff $X_a \backslash S \subseteq X_a \backslash T$, i.e., iff $X_a \cap I \subseteq X_a \cap J$. Thus—noting that X_a is just the set of nonzero elements $x \leq a$—we have

$$\Gamma_a(I) = \bigcap \{J | a' \in J \;\&\; \forall_x \leq a, x \in I \Rightarrow x \in J\}$$

But this is just to say that

$$\begin{aligned}\Gamma_a(I) &= (\{a'\} \cup (X_a \cap I)) \\ &= (\{a' \oplus x | x \leq a \;\&\; x \in I\}) \\ &= (\{(a - x)' \, | x \leq a \;\&\; x \in I\}) \\ &= (\{y | y' \leq a \;\&\; a - y' \in I\})\end{aligned}$$

(b) In terms of filters, we have

$$\begin{aligned}\Delta_a(F) &= \langle\{(a' \oplus x)' \, | x \leq a \;\&\; x \in I\}\rangle \\ &= \langle\{(a - x) | x \leq a \;\&\; x \in I\}\rangle \\ &= \langle\{y \leq a | (a - y) \in I\}\rangle \\ &= \langle\{y \leq a | (a - y)' = y \oplus a' \in F\}\rangle \quad \blacksquare\end{aligned}$$

Remark. Notice that conditions (a) and (b) of Lemma 3 make perfectly good sense if L is replaced by an arbitrary effect algebra. In that case, we take them as *defining* the maps Γ and Δ.

Bennet and Foulis [1] introduce (for any effect algebra) the quantity

$$\nabla(a, b) := \{x \leq a | b \leq x \oplus a'\}$$

They then defined their generalized Sasaki projection $\Phi(a, b)$ to be the set of all minimal elements of $\nabla(a, b)$ (if any). If L is an OML, there is a unique minimal element, namely $\phi(a, b)$.

Theorem 3. Let L be an OMP. Then, for all $b \in L$, $\Delta_a(\langle b\rangle) = \langle\nabla(a, b)\rangle$. If L satisfies the descending chain condition (in particular, if L is finite), then $\Delta_a(\langle b\rangle) = \langle\Phi(a, b)\rangle$.

Proof. An orthoalgebra L is an OMP iff the filter $\langle b\rangle$ generated by $b \in L$ coincides with the principal order-filter $\{y \in L | y \geq b\}$. Part (b) of Lemma 2 then yields $\Delta_a(\langle b\rangle) = \langle\{y \leq a | b \leq y \oplus a'\}\rangle$, i.e., $\Delta_a(\langle b\rangle) = \langle\nabla(a, b)\rangle =$

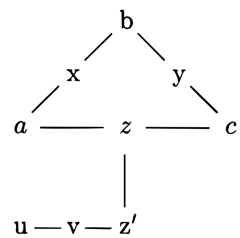

Fig. 2.

$\langle\Phi(a, b)\rangle$. If L satisfies the descending chain condition, then every element of $\nabla(a, b)$ lies above a minimal element, so $\langle\nabla(a, b)\rangle = \langle\Phi(a, b)\rangle$. ■

Notice that this supplies the proof of Theorem 2, since $\langle\nabla(a, b)\rangle = \langle\phi(a, b)\rangle$ for an OML.

As the following example shows, $\Delta_a(\langle b\rangle)$ and $\langle\nabla(a, b)\rangle$ need not coincide if L is not an OMP.

Example 3. Let $\mathfrak{A} = \{\{a, x, b\}, \{b, y, c\}, \{c, z, a\}, \{z', u, v\}\}$, as illustrated in Fig. 2. Identifying outcomes with the corresponding propositions in L, let $p = u \oplus z'$. The only elements $x \leq p$ in L are 0, u, z', and p itself. The corresponding elements $x \oplus p' = x \oplus v$ are v, $u \oplus v = z$, $z' \oplus v = u$, and 1. Of these, only 1 lies above b. Thus, $\nabla(p, b) = \{1\}$. On the other hand, $\langle b\rangle$ includes z, so, as $u \oplus v = z$, $u \in \{x \leq p | x \oplus p' \in \langle b\rangle\}$. Thus, $u \in \Delta_p(\langle b\rangle)$. (Indeed, a little further reflection shows that in this example, $\Delta_p(\langle b\rangle) = \langle u\rangle$.)

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